

# Oscillatory properties of solutions of the fourth order difference equations with quasidifferences

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**Abstract.** A class of fourth-order neutral type difference equations with quasidifferences and deviating arguments is considered. Our approach is based on studying the considered equation as a system of a four-dimensional difference system. The sufficient conditions under which the considered equation has no quickly oscillatory solutions are given. Finally, the sufficient conditions under which the equation is almost oscillatory are presented.

**Keywords:** Fourth-order difference equation, neutral type, quickly oscillatory solutions, almost oscillatory.

**AMS Subject classification:** 39A21, 39A10

## 1 Introduction

In this article, we consider a class of fourth-order nonlinear difference equations of the form:

$$\Delta \left\{ a_n \left[ \Delta \left( b_n \left( \Delta (c_n (\Delta (x_n + p_n x_{n-\delta}))^\gamma \right)^\beta \right)^\alpha \right] \right\} + d_n f(x_{n-\tau}) = 0 \quad (1)$$

where  $\alpha, \beta$  and  $\gamma$  are the ratios of odd positive integers, integers  $\tau, \delta$  are deviating arguments,  $\tau \neq \min\{-4, \delta - 4\}$ . Moreover,  $(p_n)$  is a real sequence,  $(d_n)$  is of one sign, and  $(a_n), (b_n), (c_n)$  are positive real sequences defined for  $n \in \mathbb{N}_0 = \{n_0, n_0 + 1, \dots\}$ ,  $n_0 \in \mathbb{N} = \{0, 1, 2, \dots\}$ ,  $n_0 \geq \max\{1, \delta, \tau\}$ . Function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  denotes the set of real numbers. Here  $\Delta$  is the forward difference operator defined for any real sequence  $(y_n)$  by  $\Delta y_n = y_{n+1} - y_n$ . If  $\tau > \min\{-4, \delta - 4\}$ , then obviously there exists a solution of equation (1) that can be found recursively for given initial conditions. If  $\tau < \min\{-4, \delta - 4\}$ , then in order to ensure the existence of a unique solution

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of equation (1) for given initial conditions, we assume in this paper that the function  $f$  is invertible.

In the last few years, great attention has been paid to the study of fourth-order nonlinear difference equations. It is interesting to extend the known oscillation criteria for a larger class of fourth order nonlinear difference equations with quasidifferences. The oscillatory and asymptotic properties of fourth order nonlinear difference equations were investigated, among many others, by: Agarwal, Grace and Manojlović [3, 4], Došlá and Krejčová [5], Migda, Musielak, Popena, Schmeidel and Szmanda [8, 9, 10, 11, 12, 13, 14], Smith and Taylor [15], and Arockiasamy, Dhanasekaran, Graef, Pandian and Thandapani [16, 17, 18], and references therein.

The background for difference equations can be found in the well known monographs, see, for example: Agarwal, Bohner, Grace and O'Regan [2], Agarwal [1], and Kelly and Peterson [6].

Set

$$z_n = x_n + p_n x_{n-\delta}. \quad (2)$$

The sequence  $(z_n)$  is called companion sequence of a sequence  $(x_n)$  relative to  $(p_n)$ .

By a solution of equation (1) we mean a real sequence  $(x_n)$  satisfying equation (1) for  $n \in \mathbb{N}_0$ . A nontrivial solution  $(x_n)$  of (1) is called nonoscillatory if it is either eventually positive or eventually negative, and it is otherwise oscillatory. Equation (1) is called almost oscillatory if all its solutions are oscillatory or

$$\lim_{n \rightarrow \infty} x_n = 0.$$

A solution  $(x_n)$  of equation (1) is called quickly oscillatory if

$$x_n = (-1)^n q_n, \text{ where } q_n \text{ is of one sign for } n \in \mathbb{N}_0.$$

If  $p_n \equiv 0$  and  $f(x) = x^\lambda$ , then equation (1) takes the following form

$$\Delta(a_n(\Delta b_n(\Delta c_n(\Delta x_n)^\gamma)^\beta)^\alpha) + d_n x_{n+\tau}^\lambda = 0.$$

Since in our consideration  $\tau \in \mathbb{Z}$  we see that this special case of equation (1) with negative  $\tau$  was studied in [5]. The results presented in this paper have improved and generalized those obtained by Došlá and Krejčová.

We consider (1) as a four-dimensional system. If

$$y_n = c_n(\Delta z_n)^\gamma, \quad w_n = b_n(\Delta y_n)^\beta, \quad t_n = a_n(\Delta w_n)^\alpha,$$

then equation (1) can be written as the nonlinear system

$$\begin{cases} \Delta(x_n + p_n x_{n-\delta}) = C_n y_n^{\frac{1}{\gamma}}, \\ \Delta y_n = B_n w_n^{\frac{1}{\beta}}, \\ \Delta w_n = A_n t_n^{\frac{1}{\alpha}}, \\ \Delta t_n = -d_n f(x_{n-\tau}), \end{cases} \quad (3)$$

where

$$A_n = a_n^{-\frac{1}{\alpha}}, \quad B_n = b_n^{-\frac{1}{\beta}}, \quad C_n = c_n^{-\frac{1}{\gamma}}. \quad (4)$$

In this paper we study the properties of solutions of equation (1). Firstly, we show the influence of the deviating arguments  $\delta$  and  $\tau$  on the existence of quickly oscillatory solutions of (1). Next, some monotonic properties of solution of the considered equation written as system (3) are given. Finally, the sufficient conditions under which equation (1) is almost oscillatory are given. The results are illustrated by examples.

## 2 Some basic lemmas

In 2005, Migda and Migda presented the following result which will be used in the sequel (see [7], Lemma 1).

**Lemma 1.** *Let  $(x_n)$ ,  $(p_n)$  be a real sequences and  $(z_n)$  be a sequence define by (2), for  $n \geq \delta$ . Assume that  $(x_n)$  is bounded,  $\lim_{n \rightarrow \infty} z_n = l \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} p_n = p \in \mathbb{R}$ . If  $|p| \neq 1$ , then  $(x_n)$  is convergent and  $\lim_{n \rightarrow \infty} x_n = \frac{l}{1+p}$ .*

The next lemma is a simple generalization of Lemma 2 from Migda and Migda paper [7].

**Lemma 2.** *Assume that  $x: \mathbb{N} \rightarrow \mathbb{R}$  and*

$$\lim_{n \rightarrow \infty} p_n = p \text{ where } |p| < 1. \quad (5)$$

*If sequence  $(z_n)$  defined by (2) is bounded, then sequence  $(x_n)$  is bounded too.*

*Proof.* Let  $L > 0$  be such that  $|z_n| \leq L$  for  $n \geq 1$ . Set  $P := \frac{1+|p|}{2}$ . From (5), we get  $0 < P < 1$ , and there exists  $n_1 \in \mathbb{N}$  such that  $|p_n| \leq P$ , for  $n \geq n_1$ . Set

$$K = \max\{|x_{n_1}|, |x_{n_1+1}|, \dots, |x_{n_1+\delta+1}|\}. \quad (6)$$

Let  $n \geq n_1$  be an arbitrary integer. Then there exists  $m \in \mathbb{N}$  such that  $n_1 \leq n - m\delta \leq n_1 + \delta$ . From (2), we have

$$|x_n| \leq |z_n| + P|x_{n-\delta}| \leq L + P|x_{n-\delta}|.$$

Analogously

$$|x_{n-\delta}| \leq L + P|x_{n-2\delta}|,$$

and

$$|x_n| \leq L + PL + P^2|x_{n-2\delta}|.$$

After  $m$  steps, we obtain

$$|x_n| \leq L + PL + P^2L + \cdots + P^{m-1}L + P^m|x_{n-m\delta}|.$$

Using (6), we have  $|x_{n-m\delta}| \leq K$ . Since  $0 < P < 1$ , we obtain  $P^m|x_{n-m\delta}| < K$ . From the above,

$$|x_n| \leq L(1 + P + P^2 + \cdots + P^{m-1}) + K \leq K + \frac{L}{1 - P}.$$

This completes the proof.  $\square$

Note that, putting  $p_n \equiv p$  in the above Lemma, we get Lemma 2 from paper [7].

### 3 Existence of quickly oscillatory solutions

In this section necessary conditions for the existence of quickly oscillatory solution of equation (1) are presented.

**Theorem 1.** *Assume that  $p_n \geq 0$ ,  $d_n > 0$  for  $n \in N_0$ ,  $\delta$  is even, and*

$$xf(x) > 0, \text{ for } x \neq 0. \quad (7)$$

*If  $\tau$  is even, then equation (1) has no quickly oscillatory solutions with positive even terms. If  $\tau$  is odd, then equation (1) has no quickly oscillatory solutions with positive odd terms.*

*Proof.* Let

$$x_n = (-1)^n q_n \text{ be a quickly oscillatory solution of (1).} \quad (8)$$

Assume also that  $(q_n)$  is a positive sequence. Since  $\delta$  is even, we get

$$\begin{aligned}\Delta z_n &= \Delta(x_n + p_n x_{n-\delta}) = x_{n+1} + p_{n+1} x_{n+1-\delta} - x_n - p_n x_{n-\delta} \\ &= (-1)^{n+1} (q_{n+1} + q_n + p_{n+1} q_{n-\delta+1} + p_n q_{n-\delta}).\end{aligned}$$

Hence  $(\Delta z_n)$  is a quickly oscillatory sequence and we can write  $\Delta z_n = (-1)^{n+1} s_n$  where  $s_n = q_{n+1} + q_n + p_{n+1} q_{n-\delta+1} + p_n q_{n-\delta} > 0$ . From the first equation of system (3) we have

$$y_n = \left( \frac{\Delta z_n}{C_n} \right)^\gamma = (-1)^{n+1} r_n,$$

where  $r_n = \left( \frac{s_n}{C_n} \right)^\gamma > 0$ . From the second equation of (3) we get

$$w_n = \left( \frac{\Delta y_n}{B_n} \right)^\beta = (-1)^n l_n,$$

where  $l_n = \left( \frac{r_{n+1}}{B_n} + \frac{r_n}{B_n} \right)^\beta > 0$ . Repeating argument, we get from the third equation of (3) the following equality

$$t_n = \left( \frac{\Delta w_n}{A_n} \right)^\alpha = (-1)^{n+1} g_n,$$

where  $g_n = \left( \frac{l_{n+1}}{A_n} + \frac{l_n}{A_n} \right)^\alpha > 0$ . Consequently, from the fourth equation we have

$$\Delta t_n = (-1)^n (g_{n+1} + g_n) = -d_n f(x_{n-\tau}).$$

Hence

$$(-1)^{n+1} (g_{n+1} + g_n) = d_n f(x_{n-\tau}). \quad (9)$$

By (7), we have  $x_{n-\tau} f(x_{n-\tau}) > 0$ . From (8) we see that  $x_{n-\tau} = (-1)^{n-\tau} q_{n-\tau}$ . Since  $\tau$  is even we get  $f(x_{n-\tau})$  is positive for even  $n$ . The left-hand side of equality (9) is negative for even  $n$ , whereas the right-hand side is positive and vice versa. This contradiction ended the proof in the case of even  $\tau$ .

For odd  $\tau$  the proof is analogous and hence is omitted.  $\square$

**Remark 1.** Assume that  $p_n \geq 0$ ,  $d_n < 0$  for  $n \in N_0$ ,  $\delta$  is even, and condition (7) is satisfied. If  $\tau$  is even, then equation (1) has no quickly oscillatory solutions with positive odd terms. If  $\tau$  is odd, then equation (1) has no quickly oscillatory solutions with positive even terms.

The following examples illustrate Theorem 1. Here  $\delta$  is even and equation of the form (1) has a quickly oscillatory solution with positive even terms.

**Example 1.** Consider the equation

$$\Delta^2 \left( \Delta^2 \left( x_n + \frac{1}{2^n} x_{n-2\lambda} \right) \right)^\beta + d_n \operatorname{sgn}(x_{n-\tau}) = 0, \quad (10)$$

where  $\tau, \lambda$  are some positive integers, and  $\tau$  is odd,

$$d_n = (2^{n+2}3^2 + 2^{2-2\lambda})^\beta + 2(2^{n+1}3^2 + 2^{2-2\lambda})^\beta + (2^n3^2 + 2^{2-2\lambda})^\beta > 0,$$

and  $f(x) = \operatorname{sgn} x$ . Equation (10) has a quickly oscillatory solution  $x_n = (-1)^n 2^n$ .

**Example 2.** Consider the equation

$$\Delta^2 \left( \Delta^2 \left( x_n + \frac{1}{3^n} x_{n-2\lambda} \right) \right)^\beta + d_n x_{n-\tau} = 0 \quad (11)$$

where  $\lambda$  is some positive integer,  $\tau$  is an odd integer and

$$d_n = \left( 4 + \frac{4^2}{3^{n+4}} \right)^\beta + 2 \left( 4 + \frac{4^2}{3^{n+3}} \right)^\beta + \left( 4 + \frac{4^2}{3^{n+2}} \right)^\beta > 0$$

and  $f(x) = x$ . Equation (11) has a quickly oscillatory solution  $x_n = (-1)^n$ .

## 4 Almost oscillatory property

In this section we assume that there exists a finite limit of sequence  $(p_n)$ . After some lemmas concerning the behavior of solution of system (3), the sufficient conditions under which equation (1) is almost oscillatory are presented.

**Lemma 3.** Assume that conditions (5) and (7) are satisfied. If  $(x, y, w, t)$  is a solution of system (3) with bounded first component and such that one of its components is of one sign, then one of the following two cases hold:

- 1) all its components are of one sign for enough large  $n$ ;
- 2) sequence  $(y_n)$  is of one sign and  $\lim_{n \rightarrow \infty} x_n = 0$ .

*Proof.* Firstly, we assume that sequence  $(x_n)$  is positive. (For negative  $(x_n)$  the proof is analogous and hence omitted.) Since  $(d_n)$  is of one sign for  $n \geq n_0$ , and by (7), we have that  $(\Delta t_n)$  is also of one sign. It means that if  $\Delta t_n > 0$ , then sequence  $(t_n)$  is increasing, and if  $\Delta t_n < 0$ , then sequence

$(t_n)$  is decreasing. In both cases sequence  $(t_n)$  is of one sign for large  $n$ . This implies, by the third equation of system (3), that  $\Delta w_n$  is of one sign for large  $n$ . Repeating the same arguments as before, we obtain that sequence  $(w_n)$  is of one sign, and consequently  $(y_n)$  is of one sign for large  $n$ . Hence, condition 1) of the thesis is fulfilled.

Next, we assume that  $(y_n)$  is of one sign for  $n \in \mathbb{N}_0$ . By the first equation of system (3), we get that sequence  $(z_n)$  defined by (2) is monotonic for large  $n$ . Hence  $\lim_{n \rightarrow \infty} z_n$  exists. Let us consider three possible cases:

(i)

$$\lim_{n \rightarrow \infty} z_n = \pm\infty,$$

(ii)

$$\lim_{n \rightarrow \infty} z_n = l \neq 0,$$

(iii)

$$\lim_{n \rightarrow \infty} z_n = 0.$$

Case (i). If  $(z_n)$  is unbounded, then  $(x_n)$  is unbounded too. By assumption of the Lemma this case is excluded.

Case (ii). Since sequence  $(z_n)$  is bounded, by condition (5) and Lemma 2, sequence  $(x_n)$  is bounded too. By Lemma 1, we have  $\lim_{n \rightarrow \infty} x_n = \frac{l}{1+p} \neq 0$ . It means that sequence  $(x_n)$  is of one sign for large  $n$ .

Case (iii). By Lemmas 1 and 2, we get  $\lim_{n \rightarrow \infty} x_n = 0$ . It means that condition 2) of the thesis is satisfied.

Assuming that  $(t_n)$  or  $(w_n)$  is of one sign, on virtue of analogous arguments as the above, we also get the thesis.  $\square$

**Corollary 1.** *Assume that (5) and (7) are satisfied. If  $(x, y, w, t)$  is a solution of system (3) with bounded first component and such that one of its components is of one sign, then there exists limit of sequence  $(x_n)$  and exactly one of the following two cases are held:*

$$1) \lim_{n \rightarrow \infty} x_n \neq 0,$$

*and sequences  $y$ ,  $w$  and  $t$  are monotonic for enough large  $n$ , or*

$$2) \text{ sequence } (y_n) \text{ is of one sign and } \lim_{n \rightarrow \infty} x_n = 0.$$

**Corollary 2.** Assume that conditions (5) and (7) are satisfied, and

$$\sum_{n=n_0}^{\infty} A_n = \sum_{n=n_0}^{\infty} B_n = \sum_{n=n_0}^{\infty} C_n = \infty. \quad (12)$$

If  $(x, y, w, t)$  is a solution of system (3) such that

$$\lim_{n \rightarrow \infty} x_n \in \mathbb{R}, \quad (13)$$

then

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} t_n = 0.$$

*Proof.* Since conditions (5) and (13) hold,  $\lim_{n \rightarrow \infty} z_n$  is finite. From the first equation of system (3), we get

$$z_n = z_{n_0} + \sum_{i=n_0}^{n-1} C_i y_i^{\frac{1}{\gamma}}. \quad (14)$$

Without loss the generality, for the sake of contradiction, assume that

$\lim_{n \rightarrow \infty} y_n > 0$ . Hence  $\lim_{n \rightarrow \infty} y_n^{\frac{1}{\gamma}} > 0$ . Since  $C_n$  is positive, letting  $n$  to infinity, the right hand side of equality (14) tends to infinity whereas the left hand side has a finite limit. It follows that  $\lim_{n \rightarrow \infty} y_n = 0$ . In the similar manner the remaining part of the thesis is obtained.  $\square$

**Theorem 2.** If conditions (5), (7) and (12) are satisfied,  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous function, and series

$$\sum_{i=1}^{\infty} d_i \text{ is divergent,} \quad (15)$$

then any bounded solution of equation (1) is almost oscillatory.

*Proof.* For the sake of contradiction assume that equation (1) has a nonoscillatory bounded solution which does not approach zero. Without loss of generality we assume that  $x_n > 0$  for large  $n$ . From the above and Corollary 1, we have that finite limit  $\lim_{n \rightarrow \infty} x_n$  exists. Set  $\lim_{n \rightarrow \infty} x_n = c \in (0, \infty)$ . So, by (7), we have  $f(c) > 0$ . Then there exists a positive integer  $n_1$  such that  $f(x_{n-\tau}) \geq \frac{f(c)}{2}$  for  $n \geq n_1$ . This implies, that: if  $(d_n)$  is a positive sequence, then

$$\sum_{i=n_1}^{\infty} d_i f(x_{i-\tau}) \geq \frac{f(c)}{2} \sum_{i=n_1}^{\infty} d_i = +\infty;$$



if  $(d_n)$  is a negative sequence, then

$$\sum_{i=n_1}^{\infty} d_i f(x_{i-\tau}) \leq \frac{f(c)}{2} \sum_{i=n_1}^{\infty} d_i = -\infty.$$

Summing the last equation of system (3) from 1 into  $n-1$ , we have

$$t_n - t_1 = - \sum_{i=1}^{n-1} d_i f(x_{i-\tau}).$$

By Corollary 2, we get that sequence  $(t_n)$  tends to zero as  $n$  tends to  $\infty$ . Hence, letting  $n$  to  $\infty$ , we obtain

$$t_1 = \sum_{i=1}^{\infty} d_i f(x_{i-\tau}).$$

The left-hand side of the above equation is a constant whereas the right-hand side is  $-\infty$  or  $+\infty$ . This contradiction ended the proof.  $\square$

The following two examples show equations of type (1) the coefficients of which fulfill the assumptions of Theorem 2 and both of them have an almost oscillatory solution. The first one has an asymptotically zero solution, and the second one has an oscillatory solution.

**Example 3.** Consider the equation

$$\Delta \left( n \Delta^3 \left( x_n + \frac{1}{4} x_{n-2} \right) \right) + (1-n) x_{n+3} = 0.$$

The above equation has solution  $x_n = -\frac{1}{2^n}$  which tends to zero.

**Example 4.** Consider the equation

$$\Delta \left( n \Delta^3 \left( x_n + \frac{1}{4} x_{n-2} \right) \right) + 10(2n+1) x_{n+3} = 0.$$

This equation has oscillatory solution  $x_n = \frac{(-1)^n}{10}$ .

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